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The Luttinger model, Thirring strings, and enhanced internal symmetries

Uwe K Albertin†, Michael G Cleveland and Tim M Pritchett
Department of Physics, University of California, Berkeley, CA 94720, USA

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Abstract. The standard solution of the Luttinger model is compared with recent formulations of the Thirring model used in describing Thirring strings. These models may in turn be described by compactified boson models of particular radii. An interesting discrepancy is noted between the apparent radius of the bosons in the Luttinger model and those in the Thirring model. This difference manifests itself in different conformal weights for the fermion fields. We resolve this discrepancy by performing an additional rescaling of the Luttinger model bosons. We then examine the enhanced symmetry points that occur in the boson theory at particular values of the radii. In particular, for the Thirring model with two families of spinors, which corresponds to a conformal theory with central extension $c=2$, one finds an enhanced $SU(3)_L \times SU(3)_R$ symmetry for certain values of the boson radii. This theory is equivalent to the Luttinger model with spin. By examining the response functions for the Luttinger model it is found that the point of $SU(3)_L \times SU(3)_R$ symmetry lies on the Luttinger model phase transition lines.

1. Introduction

The Luttinger model [1] has been used extensively as a model for the one-dimensional electron gas. Because of the assumption of linear dispersion, the model is equivalent to a number of field theory models for particular choices of the coupling constants. In this paper we shall examine the relation between this model and recent treatments of the Thirring model utilised in describing Thirring strings [2]. Solutions of both models depend upon boson-fermion equivalence in one spatial dimension through the construction of collective operators that obey canonical commutation relations. The appropriate collective operators for the single-component Thirring model are the current operators that arise from the $U(1)_L \times U(1)_R$ internal symmetry, and solution of the model follows from the treatment of Dell Antonio *et al* [3]. The equivalent operators in the spinless Luttinger model are the charge density operators for right- and left-moving fermions first used by Tomonaga [4]. Such bosonisation treatments yield quadratic Hamiltonians which may be solved exactly.

A critical role is played by the charge operators in both models. The total Hilbert space may be viewed as the tensor product of boson and fermion states. States of a particular fermion number and zero boson number may be used as vacua for the construction of various boson sectors of the problem, and the contribution to the energy from the charge operators will depend upon the charges of these vacua. It is this additional contribution to the energy that leads to the equivalence of the Thirring

† Current address: Department of Physics, Columbia Univeristy, New York, NY 10027, USA.

model with compactified boson models, i.e. models where the boson fields are angular variables taking values in a circle or torus [2, 5]. Level matching then leads to a relation between the radius of the boson and the strength of the coupling constant in the Thirring model. It is an interesting feature of the resulting boson models that they display enhanced internal symmetries for particular values of the boson radii. This is most easily seen by examining the energy levels of the system and by noting that additional degeneracies exist for particular values of the radii. For the Thirring model with a single complex fermion field it has been known for some time that the $U(1)_L \times U(1)_R$ symmetry may be enhanced to $SU(2)_L \times SU(2)_R$ at the boson radius $r = (\frac{1}{2})^{1/2}$. For the Thirring model with N fermions more recent treatments by Chang and Kumar [6] reveal an $SO(2N)_L \times SO(2N)_R$ symmetry for the free theory which is broken to a $U(N)_L \times U(N)_R$ symmetry for the interacting theory. At the particular values of the radii $r_1 = (1/N)^{1/2}$, $r_2 \dots r_N = 1$ this symmetry is enhanced to an $SU(N+1)_L \times SU(N+1)_R$ symmetry where the model becomes equivalent to a Wess-Zumino-Witten model with structure group $SU(N+1)$ [7].

In this paper we examine the relation between the Luttinger model and the above models. We are particularly interested in the effective radius of the Luttinger model when written in terms of compactified bosons, and whether there is any relation between the phase structure of the Luttinger model with spin and the points of enhanced symmetry noted above. We find in the spinless case that the apparent boson radius of the Luttinger model, which may be used to calculate conformal weights and hence correlation functions, does not match the Thirring model radius. We resolve this discrepancy by an additional rescaling of the current operators. We then examine the case with spin. Since the Luttinger model with spin has two sets of fermions, one for spin up and one for spin down, the relevant enhanced symmetry is $SU(3)_L \times SU(3)_R$, and the theory is conformal invariant with central extension $c = 2$. We then examine the response functions for the Luttinger model which have been treated extensively by Solyom [8]. Singularities in the response functions are indicative of instabilities in the system which cause the formation of a particular phase as the coupling constants are varied at zero temperature. We find the interesting result that the $SU(3)_R \times SU(3)_L$ enhanced symmetry point when considered in coupling constant space lies on the phase transition lines for the Luttinger model. The paper is organised as follows. In section 2 we introduce the Luttinger model and review its standard solution. In section 3 we examine the Thirring model and its solution as given by Bagger *et al* [2]. We examine the spinless case first and then treat the case with spin in section 4.

2. The spinless Luttinger model

The Luttinger model [1] is an approximate model for the one-dimensional electron gas in a box of length L. We shall denote space and time variables by σ and τ respectively. The free Hamiltonian is obtained by linearising the dispersion relation in a small region about the Fermi points, and is given by [4]

$$H_0 = \sum_k v_f k a_k^\dagger a_k - \sum_k v_f k b_k^\dagger b_k. \tag{1}$$

The operators a correspond to right-moving particles, the b to left-moving particles. The corresponding operators in real space will be labelled ψ_R and ψ_L respectively. These operators anticommute with one another and satisfy canonical anticommutation relations among themselves. In this Hamiltonian we have implicitly done the field redefinitions $\psi_R \to \exp[-ik_F(v_F\tau-\sigma)]\psi_R$ and $\psi_L \to \exp[-ik_F(v_F\tau+\sigma)]\psi_L$ which bring the Fermi points to the origin. Normal ordering is defined relative to the ground state where the left-moving fermion states are filled for $k>0$ while the right-moving states are filled for $k<0$.

The inverse Fourier transform may be written

$$H_0 = v_F \int d\sigma\, \psi_R^\dagger\left(-i\frac{\partial}{\partial\sigma}\right)\psi_R + v_F \int d\sigma\, \psi_L^\dagger\left(i\frac{\partial}{\partial\sigma}\right)\psi_L. \tag{2}$$

The interaction for the Luttinger model follows from the many-body interaction by setting the Fourier transformed couplings to be constants, and by neglecting umklapp and large momentum transfer terms [8]. For the spinless model this may be written as [9]

$$H_{int} = (g_1/2)\int d\sigma\,(\psi_L^\dagger\psi_L\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R\psi_R^\dagger\psi_R) + g_2\int d\sigma\,\psi_R^\dagger\psi_R\psi_L^\dagger\psi_L. \tag{3}$$

The first term serves to renormalise the Fermi velocity when the bosonisation treatment is envoked, and we shall consider for the remainder of the paper only the g_2 coupling which corresponds to the Thirring coupling. The action for the total Hamiltonian then becomes with $v_F=1$

$$S = \int d\tau\, d\sigma\, \psi_R^\dagger\left(i\frac{\partial}{\partial\tau}+i\frac{\partial}{\partial\sigma}\right)\psi_R + \int d\tau\, d\sigma\, \psi_L^\dagger\left(i\frac{\partial}{\partial\tau}-i\frac{\partial}{\partial\sigma}\right)\psi_L - g_2\int d\sigma\, d\tau\,\psi_R^\dagger\psi_R\psi_L^\dagger\psi_L. \tag{4}$$

Introducing light cone coordinates

$$u=\tau+\sigma \qquad v=\tau-\sigma$$

allows this to be written as

$$S = 2\int d\tau\, d\sigma\,[i\psi_L^\dagger\partial_v\psi_L + i\psi_R^\dagger\partial_u\psi_R - (g_2/2)\psi_L^\dagger\psi_L\psi_R^\dagger\psi_R] \tag{5}$$

which corresponds to the action of the Thirring model [2]. The Luttinger model is solved by Fourier transforming the density operators $\rho_{R,L} = \psi_{R,L}^\dagger\psi_{R,L}$. The resulting operators satisfy the commutation relations

$$[\rho_R(k), \rho_R(-k')] = (-kL/2\pi)\delta_{k,k'} \qquad [\rho_L(k), \rho_L(-k')] = (kL/2\pi)\delta_{k,k'} \tag{6}$$

These commutation relations are satisfied due to the presence of the Fermi sea. The operators

$$\begin{aligned} d_{Rk}^\dagger &= (2\pi/kL)^{1/2}\rho_R(k) \qquad & d_{Rk} &= (2\pi/kL)^{1/2}\rho_R(-k) \\ d_{Lk}^\dagger &= (2\pi/kL)^{1/2}\rho_L(-k) \qquad & d_{Lk} &= (2\pi/kL)^{1/2}\rho_L(k) \end{aligned} \tag{7}$$

then obey canonical commutation relations for $k>0$. The interacting Hamiltonian may be written in terms of the boson operators as

$$H = \sum_{k>0} k[d_{Rk}^\dagger d_{Rk} + d_{Lk}^\dagger d_{Lk}] + (g_2/2\pi)k[d_{Rk}^\dagger d_{Lk}^\dagger + d_{Rk}d_{Lk}] \tag{8}$$

as long as we neglect zero-mode contributions. The contribution due to zero modes may be given as follows. Consider a ground state with particular fermion numbers n_L, n_R. It is readily shown following the treatment of Haldane [10] that the energy of this state is $(\pi/L)(n_L^2+n_R^2)$. Since it is an eigenstate of the charge operators $\rho_{0R,L}$ we see that the contribution to the energy due to the charges from the free Hamiltonian may be written in terms of the operators as $(\pi/L)(\rho_{0L}^2+\rho_{0R}^2)$. The contribution from the interaction Hamiltonian may be found by isolating the zero modes in the interaction. In terms of operators we may then add a zero-mode contribution yielding a full Hamiltonian

$$H=\sum_{k>0} k[d^\dagger_{Rk}d_{Rk}+d^\dagger_{Lk}d_{Lk}]+(g_2/2\pi)k[d^\dagger_{Rk}d^\dagger_{Lk}+d_{Rk}d_{Lk}]$$
$$+(\pi/L)(\rho_{0L}^2+\rho_{0R}^2)+(g_2/L)\rho_{0R}\rho_{0L}. \qquad (9)$$

If we work in a box of length π and set $h=g_2/4\pi$ we obtain the Hamiltonian

$$H/2=\sum_{n>0} n(d^\dagger_{nL}d_{nL}+d^\dagger_{nR}d_{nR})+\sum_{n>0} 2hn(d_{nL}d_{nR}+d^\dagger_{nL}d^\dagger_{nR})+\tfrac{1}{2}(\rho_{0L}^2+\rho_{0R}^2)+2h\rho_{0L}\rho_{0R} \qquad (10)$$

where n is an integer. This Hamiltonian is readily diagonalised using a Bogoliubov transformation. We define new operators according to

$$\hat{d}_{nL}=c(\lambda)d_{nL}+s(\lambda)d^\dagger_{nR} \qquad \hat{d}_{nR}=c(\lambda)d_{nR}+s(\lambda)d^\dagger_{nL} \qquad (11)$$

where $c(\lambda)=\cosh(\lambda)$ and $s(\lambda)=\sinh(\lambda)$. The charge operators obey similar relations so that the charges are altered when using this transformation. Expressing the Hamiltonian in terms of the new operators yields

$$H/2=(1-4h^2)^{1/2}H'=[1/\cosh(2\lambda)]H'$$
$$H'=\sum_{n>0} n(\hat{d}^\dagger_{nL}\hat{d}_{nL}+\hat{d}^\dagger_{nR}\hat{d}_{nR})+\tfrac{1}{2}\hat{\rho}_{0L}^2+\tfrac{1}{2}\hat{\rho}_{0R}^2 \qquad (12)$$

and

$$\tanh(2\lambda)=2h. \qquad (13)$$

Hence if we restore the Fermi velocity we see that this Hamiltonian appears to describe a free-boson theory with a renormalised Fermi velocity given by $\hat{v}_F=v_F/\cosh(2\lambda)$. The velocity is indeed rescaled since the total momentum given by

$$P=\sum_{k>0} k[d^\dagger_{kR}d_{kR}-d^\dagger_{kL}d_{kL}] \qquad (14)$$

remains unchanged under the Bogoliubov rotation. This essentially completes the solution of the Luttinger model. We remark that one should be cautious in this rescaling, however, because the scaling factor is not absorbed in the zero-mode piece of the Hamiltonian, which does not depend upon the Fermi velocity. We shall examine this point further in the next section.

3. Relation to field theory

We saw in the previous section how the Luttinger model could be mapped onto the Thirring model. In this section we examine the solution of the Thirring model given by Bagger *et al* [2] and its relation to the solution of the previous section. As pointed out in [2], one must be careful to distinguish between fields in the interaction picture and fully interacting fields in the Heisenberg picture. A Bogoliubov rotation on the boson operators may be viewed as relating these two sets of fields, although as we shall see this rotation is not the one that appeared in the previous section. With this in mind we shall denote fully interacting operators in the Heisenberg picture with an additional tilde, and operators in the interaction picture without the tilde, remembering to distinguish them from the hatted operators of the last section. Hence if U denotes the unitary operator that implements the Bogoliubov rotation, then for any operator $\tilde{O} = UOU^\dagger$. We use the conventions in [2] and write the action for the Heisenberg picture fields as

$$\tilde{S} = \frac{1}{\pi} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\mathrm{i}\tilde{\psi}_L^\dagger \partial_v \tilde{\psi}_L + \mathrm{i}\tilde{\psi}_R^\dagger \partial_u \tilde{\psi}_R - \tilde{h}\tilde{\psi}_R^\dagger \tilde{\psi}_L \tilde{\psi}_R^\dagger \tilde{\psi}_R). \tag{15}$$

This model is conformal invariant with central extension $c = 1$. The corresponding action in the interaction picture may be written without the tildes, and we note that the action in (5) corresponds to this action by rescaling $\psi \to \psi/(2\pi)^{1/2}$ and by taking $h = g_2/4\pi$. To solve the model it is convenient to bosonise in terms of the currents. This theory has a local $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry given by

$$\tilde{\psi}_L(u, v) \to \exp(\mathrm{i}f(u))\tilde{\psi}_L(u, v) \qquad \tilde{\psi}_R(u, v) \to \exp(\mathrm{i}g(v))\tilde{\psi}_R(u, v) \tag{16}$$

for which the conserved currents are given by

$$\tilde{J}_L = (1/2\pi) : \tilde{\psi}_L^\dagger \tilde{\psi}_L : \qquad \tilde{J}_R = (1/2\pi) : \tilde{\psi}_R^\dagger \tilde{\psi}_R : . \tag{17}$$

The normal ordering here is with respect to the Fermi fields. The analogous interaction picture currents correspond essentially to the density operators of the previous section.

With respect to the interaction picture currents the Fermi fields behave like free fields. Hence the fields have charges -1 with respect to the interaction picture currents and we may write the following commutation relations:

$$\begin{aligned} &[J_L(u), \tilde{\psi}_L(u', v')] = -\tilde{\psi}_L(u', v')\delta(u - u') \\ &[J_R(v), \tilde{\psi}_R(u', v')] = -\tilde{\psi}_R(u', v')\delta(v - v') \\ &[J_L, \tilde{\psi}_R] = 0 \qquad [J_R, \tilde{\psi}_L] = 0. \end{aligned} \tag{18}$$

With respect to the Heisenberg picture currents the fields have different vacuum charges, however, which leads to commutation relations of the form

$$\begin{aligned} &[\tilde{J}_L(u), \tilde{\psi}_L(u', v')] = (-a/2)\tilde{\psi}_L(u', v')\delta(u - u') \\ &[\tilde{J}_R(v), \tilde{\psi}_L(u', v')] = (-b/2)\tilde{\psi}_L(u', v')\delta(v - v') \\ &[\tilde{J}_R(v), \tilde{\psi}_R(u', v')] = (-c/2)\tilde{\psi}_R(u', v')\delta(v - v') \\ &[\tilde{J}_L(u), \tilde{\psi}_R(u', v')] = (-d/2)\tilde{\psi}_R(u', v')\delta(u - u') \end{aligned} \tag{19}$$

where the parameters a, b, c, d are to be determined. The currents $\tilde{J}$ and J are related by a Bogoliubov rotation as

$$\tilde{J}_R = c(\lambda)J_R + s(\lambda)J_L \qquad \tilde{J}_L = c(\lambda)J_L + s(\lambda)J_R. \tag{20}$$

This is essentially a Bogoliubov rotation of the type used in (11) although we have not yet fixed the parameter λ in this context. The commutation relations for the currents are

$$\begin{aligned} &[J_L(u), J_L(u')] = [\tilde{J}_L(u), \tilde{J}_L(u')] = (\mathrm{i}/2\pi)\delta'(u - u') \\ &[J_R(v), J_R(v')] = [\tilde{J}_R(v), \tilde{J}_R(v')] = (\mathrm{i}/2\pi)\delta'(v - v') \end{aligned} \tag{21}$$

where the second relation follows from the fact that the commutation relations are invariant under the Bogoliubov rotation. If we substitute (20) into (19) we arrive at the consistency relations

$$a = c = 2\cosh(\lambda) \qquad b = d = 2\sinh(\lambda). \tag{22}$$

Note that these also satisfy

$$a^2 - b^2 = d^2 - c^2 = 4 \tag{23}$$

which is equivalent to the condition that ψ transform like a spinor under Lorentz transformations as was noted in [2]. The relations in (22) and (23) have a one-parameter set of solutions given by

$$a = c = \rho + (1/\rho) \qquad b = d = \rho - (1/\rho). \tag{24}$$

We then have the particularly simple relation between ρ and the Bogoliubov parameter λ as

$$\rho = e^{\lambda}. \tag{25}$$

We shall Fourier analyse the currents as

$$\tilde{J}_{n\mathrm{L}} = \int_0^{\pi} e^{2inu} \tilde{J}_{\mathrm{L}}(u)\, du \qquad \tilde{J}_{n\mathrm{R}} = \int_0^{\pi} e^{2inv} \tilde{J}_{\mathrm{R}}(v)\, dv \tag{26}$$

which satisfy

$$[\tilde{J}_{m\mathrm{L}}, \tilde{J}_{-m'\mathrm{L}}] = m\delta_{mm'} \qquad [\tilde{J}_{m\mathrm{R}}, \tilde{J}_{-m'\mathrm{R}}] = m\delta_{mm'} \tag{27}$$

so that we may define boson operators that obey canonical commutation relations according to

$$\tilde{d}_{n\mathrm{L}} = \frac{\tilde{J}_{n\mathrm{L}}}{n^{1/2}} \qquad \tilde{d}_{n\mathrm{R}} = \frac{\tilde{J}_{n\mathrm{R}}}{n^{1/2}} \tag{28}$$

for $n > 0$. The procedure here is entirely analogous to that of the previous section except for the following difference. Due to the use of light cone coordinates when implementing the Fourier transform, the left-moving particles here have a filled Fermi sea for $n < 0$ just as do the right-moving particles. This accounts for the lack of a minus sign in the commutation relations for $\tilde{J}_{n\mathrm{L}}$ in (27) when compared to those in (6).

The Hamiltonian for the Thirring model may be constructed using the energy-momentum tensor which is quadratic in the currents. The energy momentum tensor for the fully interacting theory is

$$\tilde{\Theta}_{\mathrm{L}}(u) = 2\pi{:}\tilde{J}_{\mathrm{L}}(u)\tilde{J}_{\mathrm{L}}(u){:} \qquad \tilde{\Theta}_{\mathrm{R}}(v) = 2\pi{:}\tilde{J}_{\mathrm{R}}(v)\tilde{J}_{\mathrm{R}}(v){:}. \tag{29}$$

The Hamiltonian may then be written quadratically in the currents as

$$\tilde{H} = \tfrac{1}{2}\int du\, \tilde{\Theta}_{\mathrm{L}}(u) + \tfrac{1}{2}\int dv\, \tilde{\Theta}_{\mathrm{R}}(v) \tag{30}$$

so that

$$\tilde{H}/2 = \tfrac{1}{2}(\tilde{J}_{0\mathrm{L}}^2 + \tilde{J}_{0\mathrm{R}}^2) + \sum_{n>0} n(\tilde{d}^{\dagger}_{n\mathrm{L}}\tilde{d}_{n\mathrm{L}} + \tilde{d}^{\dagger}_{n\mathrm{R}}\tilde{d}_{n\mathrm{R}}). \tag{31}$$

The eigenvalues of the charge operators in this case are $-an_{\mathrm{L}} - bn_{\mathrm{R}} - q_{\mathrm{L}}$ and $-bn_{\mathrm{L}} - an_{\mathrm{R}} + q_{\mathrm{R}}$ for $\tilde{J}_{\mathrm{L}0}$ and $\tilde{J}_{\mathrm{R}0}$, respectively. These follow essentially from the relations in

(19) except that we have introduced additional vacuum charges q_L and q_R. These charges determine the boundary conditions of the fermions through the usual bosonisation prescription, with $q_{L,R}$ taking the value 0 for fermions with periodic boundary conditions and $\frac{1}{2}$ for antiperiodic.

To complete the solution in terms of the currents one may consider the equation of motion corresponding to the action in (15) and compare it to the Heisenberg equation of motion obtained using the Hamiltonian in (30). The commutation relations of components of the energy-momentum tensor with the fields are obtained following the treatment of Dell Antonio *et al* [3] yielding

$$\begin{aligned}
[\tilde{\Theta}_L(u), \tilde{\psi}_L(u', v')] &= -2\pi a{:}\tilde{J}_L\tilde{\psi}_L{:}\delta(u-u') + \tfrac{1}{4}\mathrm{i}a^2\tilde{\psi}_L(u', v')\delta'(u-u') \\
[\tilde{\Theta}_R(v), \tilde{\psi}_L(u', v')] &= -2\pi b{:}\tilde{J}_R\tilde{\psi}_L{:}\delta(v-v') + \tfrac{1}{4}\mathrm{i}b^2\tilde{\psi}_L(u', v')\delta'(v-v') \\
[\tilde{\Theta}_R(v), \tilde{\psi}_R(u', v')] &= -2\pi c{:}\tilde{J}_R\tilde{\psi}_R{:}\delta(v-v') + \tfrac{1}{4}\mathrm{i}c^2\tilde{\psi}_R(u', v')\delta'(v-v') \\
[\tilde{\Theta}_L(u), \tilde{\psi}_R(u', v')] &= -2\pi d{:}\tilde{J}_L\tilde{\psi}_R{:}\delta(u-u') + \tfrac{1}{4}\mathrm{i}d^2\tilde{\psi}_R(u', v')\delta'(u-u')
\end{aligned} \tag{32}$$

from which one obtains the Heisenberg equation of motion. The result fixes the parameter λ by the condition

$$\sinh(\lambda) = \tilde{h}. \tag{33}$$

Comparing with (13) one sees that the rotation in (20) which relates the interaction picture and Heisenberg picture currents does not correspond to the rotation used in solving the Luttinger model. We shall return to this point below.

The Hamiltonian above has a spectrum that can be made to match that of the compactified boson model, which demonstrates the equivalence of these two models. The action for the boson is

$$S = \frac{1}{2\pi}\int \mathrm{d}\tau\, \mathrm{d}\sigma\, \partial_\alpha\phi\partial_\alpha\phi \tag{34}$$

where the fields take values in a circle of radius r. The field ϕ has an eigenmode expansion given by

$$\begin{aligned}
\phi(\sigma, \tau) = \phi_0 + P\tau + 2L\sigma &+ \frac{\mathrm{i}}{2}\sum_{n>0}(1/n^{1/2})\{d_{nR}\exp[-2\mathrm{i}n(\tau-\sigma)] + d_{nL}\exp[-2\mathrm{i}n(\tau+\sigma)]\} \\
&- \frac{\mathrm{i}}{2}\sum_{n>0}(1/n^{1/2})\{d^{\dagger}_{nR}\exp[2\mathrm{i}n(\tau-\sigma)] + d^{\dagger}_{nL}\exp[2\mathrm{i}n(\tau-\sigma)]\}.
\end{aligned} \tag{35}$$

Here P is the total momentum of the field which is canonically conjugate to ϕ_0 and L is the angular momentum. The wavefunction $\exp(\mathrm{i}P\phi_0)$ is required to be single valued, which restricts the eigenvalues of P to be $P = m/r$ where m is an integer. Further, one requires that the bosonic field satisfy the boundary conditions $\phi(\sigma+\pi) = \phi(\sigma) + 2\pi nr$ where n is an integer, which restricts L to take the values $L = nr$. The Hamiltonian for this system may be written

$$H = \frac{1}{2\pi}\int \mathrm{d}\sigma\, (\dot{\phi}^2 + \phi'^2) \tag{36}$$

which becomes

$$H/2 = m^2/4r^2 + n^2r^2 + \sum_n n(d^{\dagger}_{nL}d_{nL} + d^{\dagger}_{nR}d_{nR}). \tag{37}$$

Examining the expectation value of the Hamiltonian in (31) on various eigenstates of the $\tilde{J}_{0\mathrm{L,R}}$ where $q_{\mathrm{L}}=q_{\mathrm{R}}=0$, one finds that the spectrum of this Hamiltonian matches that in (37) with $n=\frac{1}{2}(n_{\mathrm{R}}+n_{\mathrm{L}})$ and $m=n_{\mathrm{R}}-n_{\mathrm{L}}$ and $\rho=r$. Hence the Thirring model with parameter ρ is equivalent to a boson theory with radius $r=\rho$. Note that the winding number n is equal to one half the total charge, while the total momentum depends upon the difference between the number of right and left movers, as it should. We remark that in terms of the non-interacting charges the Hamiltonian may be written as

$$H/2=H_{\mathrm{R}}+H_{\mathrm{L}}$$

with

$$\begin{aligned} H_{\mathrm{L}}&=\tfrac{1}{8}[(J_{0\mathrm{L}}+J_{0\mathrm{R}})r+(J_{0\mathrm{L}}-J_{0\mathrm{R}})(1/r)]^2+\sum_{n>0} n d^{\dagger}_{n\mathrm{L}}d_{n\mathrm{L}} \\ H_{\mathrm{R}}&=\tfrac{1}{8}[(J_{0\mathrm{L}}+J_{0\mathrm{R}})r-(J_{0\mathrm{L}}-J_{0\mathrm{R}})(1/r)]^2+\sum_{n>0} n d^{\dagger}_{n\mathrm{R}}d_{n\mathrm{R}}. \end{aligned} \tag{38}$$

The free-fermion theory is seen to correspond to radius $r=1$.

It is now easy to see the enhanced symmetry which occurs at the particular value of the radius $r=(\frac{1}{2})^{1/2}$. At this point the $\mathrm{U}(1)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{R}}$ symmetry is enlarged to $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$. Let us label the various vacua in the boson sectors by their charges $J_{0\mathrm{R}}$, $J_{0\mathrm{L}}$. Then to illustrate the symmetry we may examine the following nine states which are all degenerate in energy and form the carrier space for the adjoint representation of $\mathrm{SU}(2)_{\mathrm{R}}\times\mathrm{SU}(2)_{\mathrm{L}}$ [5]

$$\begin{array}{llll} d^{\dagger}_{1\mathrm{L}}d^{\dagger}_{1\mathrm{R}}|0,0\rangle & |\pm2,0\rangle & |0,\pm2\rangle & \\ d^{\dagger}_{1\mathrm{R}}|1,1\rangle & d^{\dagger}_{1\mathrm{R}}|-1,-1\rangle & d^{\dagger}_{1\mathrm{L}}|1,-1\rangle & d^{\dagger}_{1\mathrm{L}}|-1,1\rangle. \end{array} \tag{39}$$

These states are all found to have energy 2 at the value of $r=(\frac{1}{2})^{1/2}$, illustrating the enhanced symmetry.

Up until now we have been considering the fully interacting fields in the action in (15). We now consider what happens to an action written in terms of the interaction picture fields. The action is

$$S=\frac{1}{\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,(\mathrm{i}\psi^{\dagger}_{\mathrm{L}}\partial_{v}\psi_{\mathrm{L}}+\mathrm{i}\psi^{\dagger}_{\mathrm{R}}\partial_{u}\psi_{\mathrm{R}}-h\psi^{\dagger}_{\mathrm{L}}\psi_{\mathrm{L}}\psi^{\dagger}_{\mathrm{R}}\psi_{\mathrm{R}}). \tag{40}$$

In this case it is easiest to appeal directly to the bosonisation prescription and the treatment in [2]. We suppose that the kinetic term alone corresponds to a compactified boson model of radius 1. Both the interaction term and the kinetic term bosonise to kinetic boson actions so that the equivalent boson action is

$$S=(1/2\pi)(1+2h)\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\partial_{\mu}\phi\partial_{\mu}\phi.$$

Hence the boson fields must be rescaled by a factor $(1+2h)^{1/2}$. Since we started with a free-fermion theory corresponding to radius 1 we recognise this as the new radius so that we have the condition found in [2, 11]

$$r_h=\mathrm{e}^{\lambda}=(1+2h)^{1/2} \tag{41}$$

which gives the effective boson radius for the action in (40). Equating this with the expression for r_h obtained using (33), we see that the two couplings are related by the scaling $\tilde{h}/h = 1/r_h$. This amounts to a scaling of the currents, as may be seen from the equation of motion

$$\partial_v \psi_L = -2\pi i h : J_R \psi_L \tag{42}$$

whereby substituting for h and rescaling the current by $\tilde{J} = r_h J$ gives the Heisenberg picture equation of motion. Thus a rescaling of the currents (or equivalently the ϕ field) effectively changes the radius from 1 to r_h.

Having solved the Thirring model, we may now compare the solution presented in this section with that of the previous section. Since the Luttinger model may be mapped onto the Thirring model using the assumptions of the previous section, i.e. both models have the same action written in terms of interaction-picture fields, the Luttinger model solution should be equivalent to a free boson of radius $r_h = e^{\lambda} = (1+2h)^{1/2}$. If we, however, use the Bogoliubov rotation used to solve the Luttinger model we obtain the condition $r_l = e^{\lambda}$ where $\tanh(2\lambda) = 2h$. Solving for the radius yields the condition

$$r_l = \left(\frac{1+2h}{1-2h}\right)^{1/4} \tag{43}$$

which does not match the Thirring model result.

The discrepancy between these two lies in the renormalisation of the Hamiltonian in (12) that is done after the Bogoliubov transformation. This may be resolved as follows. Suppose that rather than follow the condensed matter prescription of absorbing the scaling factor $1/\cosh(2\lambda)$ into the Fermi velocity, we rescale the currents instead. This amounts to rescaling d, $d^{+} \to [\cosh(2\lambda)]^{1/2} d$, $[\cosh(2\lambda)]^{1/2} d^{+}$ and $\rho_{0\mathrm{L,R}} \to [\cosh(2\lambda)]^{1/2} \rho_{0\mathrm{R,L}}$. From the discussion above we see that the effect of this rescaling is to change the radius r_l by the additional scaling factor $[\cosh(2\lambda)]^{-1/2}$ to give the new radius

$$r_h = r_l [\cosh(2\lambda)]^{-1/2}.$$

Substituting the value from (13) yields

$$\begin{aligned} r_h &= \left(\frac{1+2h}{1-2h}\right)^{1/4} (1-4h^2)^{1/4} \\ &= (1+2h)^{1/2} \end{aligned} \tag{44}$$

which matches the interaction-picture field theory result. We may make several remarks at this point, the first being that the rotation of the previous section which diagonalises the Hamiltonian is not the one that takes us between the interaction picture and the Heisenberg picture in the sense of Bagger *et al*, in fact it takes us only part of the way there, the rest being accomplished by a further rescaling of the currents. Secondly, if we choose the condensed matter theory approach and rescale the Fermi velocity instead, we have a theory with a new light cone and different radius given by r_l instead of r_h. Bagger *et al* point out, however, that a simple rescaling of the currents cannot change the physics, and hence that the radius r_l is in some sense fictitious. Nevertheless, what we conclude from this treatment is that a theory with radius r_h looks like a theory where the Fermi velocity is rescaled and the radius appears as a different radius r_l. With this latter treatment the conformal weights of the fields will appear completely different, and this will be reflected in completely different correlation functions.

To further check this we may compute the correlation functions of the Fermi fields directly. Since the model is conformal invariant, it is easiest to use conformal field theory results [12]. All that is required to compute the correlation functions are the conformal weights of the fields. From the relations in (32) we immediately see that the conformal weights of the field ψ_L are $(a^2/8, b^2/8)$ and those of ψ_R are $(b^2/8, a^2/8)$. Once the conformal weights have been determined it is simple to write down the two-point functions. Taking $\sigma \to i\sigma$ and setting $z = \tilde{v}_F\tau + i\sigma$ yields

$$\begin{aligned}\langle\psi_L^\dagger(z,\bar z)\psi_L(z',\bar z')\rangle &= (z-z')^{-(1/8)[\rho+(1/\rho)]}(\bar z-\bar z')^{-(1/8)[\rho-(1/\rho)]}\\ &= (z-z')^{-\cosh(\lambda)^2/2}(\bar z-\bar z')^{-\sinh(\lambda)^2/2}\\ \langle\psi_R^\dagger(z,\bar z)\psi_R(z',\bar z')\rangle &= (\bar z-\bar z')^{-(1/8)[\rho+(1/\rho)]}(z-z')^{-(1/8)[\rho-(1/\rho)]}\\ &= (\bar z-\bar z')^{-\cosh(\lambda)^2/2}(z-z')^{-\sinh(\lambda)^2/2}\end{aligned} \tag{45}$$

With the Bogoliubov transformation parameter fixed by (13) the results match known correlation functions of the Luttinger model as in [10], where the complex coordinate z is obtained using the renormalised Fermi velocity $\tilde{v}_F$. For the Thirring model solution in the interaction picture, however, the complex coordinate is fixed by the original Fermi velocity and the conformal weights of the fields are fixed by the condition in (41). We see that the difference in conformal weights between the two solutions is accompanied by a change in the Fermi velocity, which is rescaled from v_F to $\tilde{v}_F$.

4. The Luttinger model with spin

The above techniques may be used for the case of particles with spin as well. In this case the fields have an additional spin index which we label i where $i=1$ corresponds to spin up and $i=2$ corresponds to spin down. The resulting multicomponent model has an action in the interaction picture given by

$$\begin{aligned}S = \frac{i}{\pi}\sum_i \int d\sigma\, d\tau \left[\psi_{Li}^\dagger \frac{\partial}{\partial v}\psi_{Li} + \psi_{Ri}^\dagger\frac{\partial}{\partial u}\psi_{Ri}\right] - \frac{1}{\pi}\sum_{ij}\int d\sigma\, d\tau\, h_{ij}(2\pi)^2 J_{Li}J_{Rj}\\ -\frac{1}{2\pi}\sum_{ij}\int d\sigma\, d\tau\, g_{ij}(2\pi)^2(J_{Li}J_{Lj}+J_{Ri}J_{Rj}).\end{aligned} \tag{46}$$

This model is again exactly solvable using bosonisation techniques as in [6, 8], and with the neglect of the term involving g_{ij} is conformal invariant with central extension $c=2$. Similar to the case of the spinless Luttinger model, the interacting Hamiltonian in this approximation may be written in boson form as

$$\begin{aligned}H/2 = \sum_i\sum_{n>0} n(d_{nLi}^\dagger d_{nLi} + d_{nRi}^\dagger d_{nRi}) + \sum_{ij}\sum_{n>0} 2nh_{ij}(d_{nLi}d_{nRj} + d_{nLi}^\dagger d_{nRj}^\dagger)\\ +\tfrac{1}{2}\sum_i (J_{0Li}^2 + J_{0Ri}^2) + \sum_{ij} 2h_{ij}J_{0Li}J_{0Rj}\end{aligned} \tag{47}$$

where we take h_{ij} to be symmetric. The Hamiltonian may be separated into spin density and charge density parts using a canonical transformation U' which acts on the currents to give new currents defined by

$$P_i = (OJ)_i \tag{48}$$

where the matrix O is given by

$$O = \frac{1}{2^{1/2}}\begin{pmatrix} 1 & 1\\ -1 & 1\end{pmatrix}. \tag{49}$$

We shall label the new boson operators by d_i' i.e.

$$d'_i = \sum_j O_{ij} d_j. \tag{50}$$

Here d'_1 corresponds to the usual charge density wave operators and d'_2 to the spin density wave operators in the treatment of Solyom [8]. This transformation diagonalises the matrix h_{ij}. We then have the diagonal matrix

$$D = OhO^{\mathrm{T}} \tag{51}$$

$$D = \begin{pmatrix} h_{11} + h_{12} & 0 \\ 0 & h_{11} - h_{12} \end{pmatrix}. \tag{52}$$

Under this transformation the Hamiltonian goes to

$$H/2 = \sum_i \sum_{n>0} n(d'^{\dagger}_{nLi} d'_{nLi} + d'^{\dagger}_{nRi} d'_{nRi}) + \sum_i \sum_{n>0} 2D_{ii} n(d'_{nLi} d'_{nRi} + d'^{\dagger}_{nLi} d'^{\dagger}_{nRi})$$
$$+ \tfrac{1}{2} \sum_i (P^2_{0Li} + P^2_{0Ri}) + \sum_i 2D_{ii} P_{0Li} P_{0Ri}. \tag{53}$$

The two spin pieces may now be diagonalised separately using Bogoliubov rotations with the parameters given by

$$\tanh(2\lambda_1) = 2(h_{11} + h_{12}) \qquad \tanh(2\lambda_2) = 2(h_{11} - h_{12}) \tag{54}$$

where the boson radii are given by $r_{l1} = \mathrm{e}^{\lambda_1}$ and $r_{l2} = \mathrm{e}^{\lambda_2}$. The Hamiltonian is then

$$H/2 = \sum_i [1/\cosh(2\lambda_i)] H_i$$
$$H_i = \sum_{n>0} n(\hat{d}'^{\dagger}_{nLi} \hat{d}'_{nLi} + \hat{d}'^{\dagger}_{nRi} \hat{d}'_{nRi}) + \tfrac{1}{2}\hat{P}^2_{0Li} + \tfrac{1}{2}\hat{P}^2_{0Ri}. \tag{55}$$

Note that the spin density ($i = 2$) and charge density ($i = 1$) sectors have different renormalised propagation velocities due to the different Bogoliubov rotations in (54).

On the other hand, we have the interaction picture Thirring model solution of the same action in (46) with $g_{ij} = 0$ in terms of two bosons where the radii are given by

$$r_{h1} = [1 + 2(h_{11} + h_{12})]^{1/2} \qquad r_{h2} = [1 + 2(h_{11} - h_{12})]^{1/2} \tag{56}$$

and the Fermi velocity is not renormalised. As in (38), the Hamiltonian may be written in terms of the non-interacting currents as

$$H/2 = H_{\mathrm{L}} + H_{\mathrm{R}}$$

with

$$H_{\mathrm{L}} = \tfrac{1}{8} \sum_i [(P_{0Li} + P_{0Ri}) r_{hi} + (P_{0Li} - P_{0Ri})(1/r_{hi})]^2 + \sum_i \sum_{n>0} n d^{\dagger}_{Li} d_{Li}$$
$$H_{\mathrm{R}} = \tfrac{1}{8} \sum_i [(P_{0Li} + P_{0Ri}) r_{hi} - (P_{0Li} - P_{0Ri})(1/r_{hi})^2 + \sum_i \sum_{n>0} n d^{\dagger}_{Ri} d_{Ri}. \tag{57}$$

We shall denote the eigenvalues of the charge operators J_{0Li} and J_{0Ri} by the vectors $-\boldsymbol{n}_{\mathrm{L}} - \boldsymbol{q}_{\mathrm{L}}$ and $-\boldsymbol{n}_{\mathrm{R}} + \boldsymbol{q}_{\mathrm{R}}$, respectively. We restrict to the same boundary conditions for all the fermions so that $q_{\mathrm{R}i} = q_{\mathrm{L}i} = q$. The values of q are $0, \frac{1}{2}$, corresponding to periodic or antiperiodic boundary conditions for the fermions. Further, we let $\boldsymbol{e}_i/(2^{1/2})$ be the

ith row vector of the matrix O and restrict to the case where $r_{h1}=r$ and $r_{h2}=1$. Then the above may be rewritten as

$$\begin{aligned}H_{\mathrm{L}}&=\tfrac{1}{16}\left((e_1\cdot n_{\mathrm{R}}+e_1\cdot n_{\mathrm{L}})r+(e_1\cdot n_{\mathrm{L}}-e_1\cdot n_{\mathrm{R}}+4q)\frac{1}{r}\right)^2\\&\quad+\tfrac{1}{2}[n_{\mathrm{L}}\cdot n_{\mathrm{L}}-\tfrac{1}{2}(e_1\cdot n_{\mathrm{L}})^2]+\sum_i\sum_{n>0}nd^{\dagger}_{\mathrm{L}i}d_{\mathrm{L}i}\\H_{\mathrm{R}}&=\tfrac{1}{16}\left((e_1\cdot n_{\mathrm{R}}+e_1\cdot n_{\mathrm{L}})r-(e_1\cdot n_{\mathrm{L}}-e_1\cdot n_{\mathrm{R}}+4q)\frac{1}{r}\right)^2\\&\quad+\tfrac{1}{2}\left(\sum_{i=1}^{2}(n_{\mathrm{R}i}-q)^2-\tfrac{1}{2}(e_1\cdot n_{\mathrm{R}}-2q)^2\right)+\sum_i\sum_{n>0}nd^{\dagger}_{\mathrm{R}i}d_{\mathrm{R}i}.\end{aligned}\tag{58}$$

Examination of the relations in (58) reveals additional degeneracies in states at the values $r=(\frac{1}{3})^{1/2}$ which correspond to various representations of $\mathrm{SU}(3)_{\mathrm{L}}\times\mathrm{SU}(3)_{\mathrm{R}}$. For example, there are eight states at energy $H_{\mathrm{L}}=1$, $H_{\mathrm{R}}=0$ corresponding to the adjoint representation of SU(3). We note that unlike the spinless case, the symmetry here appears to mix states of different q values and hence states with different fermion boundary conditions. Further details may be found in [6].

We now examine the relation of these enhanced symmetries to the phase structure of the system. At zero temperature the system has several phases that are a function of the coupling constants, and the phase portrait is determined from the response functions, which are as follows. They are all of the form

$$R=\langle T[O^{\dagger}(u,v)O(u',v')]\rangle\tag{59}$$

where T denotes time ordering. For the charge density wave response function N we have

$$O=\psi^{\dagger}_{\mathrm{L}1}\psi_{\mathrm{R}1}+\psi^{\dagger}_{\mathrm{L}2}\psi_{\mathrm{R}2}.\tag{60}$$

For the spin density wave response function χ

$$O=\psi^{\dagger}_{\mathrm{L}1}\psi_{\mathrm{R}2}\pm\psi^{\dagger}_{\mathrm{L}2}\psi_{\mathrm{R}1}.\tag{61}$$

For the singlet superconductor response function Δ_{s}

$$O=\psi_{\mathrm{L}1}\psi_{\mathrm{R}2}+\psi_{\mathrm{R}1}\psi_{\mathrm{L}2}\tag{62}$$

while for the triplet superconductor response function Δ_{t}

$$O=\psi_{\mathrm{L}1}\psi_{\mathrm{R}2}-\psi_{\mathrm{R}1}\psi_{\mathrm{L}2}\tag{63}$$

$$O_i=\psi_{\mathrm{L}i}\psi_{\mathrm{R}i}.\tag{64}$$

The second subscript here labels the spin. The phase properties of the system are determined by these functions; the particular phase that the system is in at a particular value of the coupling constants is determined by which response function is singular and has the highest inverse power behaviour in $|\sigma-\sigma'|$. In computing these functions only one term need be computed to show the singular behaviour. The terms chosen are

$$\begin{aligned}N(u,v)&\sim\langle T[\psi^{\dagger}_{\mathrm{L}1}(u,v)\psi_{\mathrm{R}1}(u,v)\psi^{\dagger}_{\mathrm{R}1}(0,0)\psi_{\mathrm{L}1}(0,0)]\rangle\\\chi(u,v)&\sim\langle T[\psi^{\dagger}_{\mathrm{L}2}(u,v)\psi_{\mathrm{R}1}(u,v)\psi^{\dagger}_{\mathrm{R}1}(0,0)\psi_{\mathrm{L}2}(0,0)]\rangle\\\Delta_{\mathrm{s}}(u,v)&\sim\langle T[\psi_{\mathrm{L}2}(u,v)\psi_{\mathrm{R}1}(u,v)\psi^{\dagger}_{\mathrm{R}1}(0,0)\psi^{\dagger}_{\mathrm{L}2}(0,0))\rangle\\\Delta_{\mathrm{t}}(u,v)&\sim\langle T[\psi_{\mathrm{L}1}(u,v)\psi_{\mathrm{R}1}(u,v)\psi^{\dagger}_{\mathrm{R}1}(0,0)\psi^{\dagger}_{\mathrm{L}1}(0,0)]\rangle.\end{aligned}\tag{65}$$

These response functions can be computed using bosonisation. In doing so it is necessary to do the transformation U' first, followed by the appropriate Bogoliubov

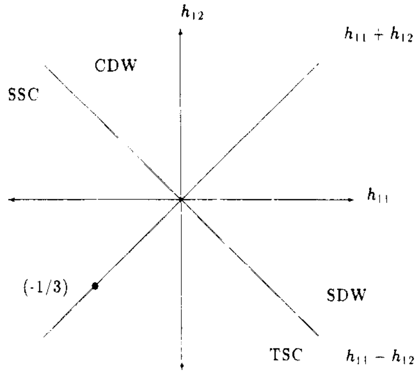

Figure 1. The phase structure of the Luttinger-Thirring model is seen here in relation to the enhanced internal symmetry. The diagonal lines intersecting the origin are the phase transition lines. The dot at $(-\frac{1}{3})$ represents the point of $SU(3)_L \times SU(3)_R$ symmetry calculated using the interaction picture Thirring model solution.

transformation. The result is

$$N \sim \left[\frac{1}{z_1\bar{z}_1}\right]^{(1/2)[\cosh(2\lambda_1)-\sinh(2\lambda_1)]}\left[\frac{1}{z_2\bar{z}_2}\right]^{(1/2)[\cosh(2\lambda_2)-\sinh(2\lambda_2)]} \tag{66}$$

$$\chi \sim \left[\frac{1}{z_1\bar{z}_1}\right]^{(1/2)[\cosh(2\lambda_1)-\sinh(2\lambda_1)]}\left[\frac{1}{z_2\bar{z}_2}\right]^{(1/2)[\cosh(2\lambda_2)+\sinh(2\lambda_2)]} \tag{67}$$

$$\Delta_s \sim \left[\frac{1}{z_1\bar{z}_1}\right]^{(1/2)[\cosh(2\lambda_1)+\sinh(2\lambda_1)]}\left[\frac{1}{z_2\bar{z}_2}\right]^{(1/2)[\cosh(2\lambda_2)-\sinh(2\lambda_2)]} \tag{68}$$

$$\Delta_t \sim \left[\frac{1}{z_1\bar{z}_1}\right]^{(1/2)[\cosh(2\lambda_1)+\sinh(2\lambda_1)]}\left[\frac{1}{z_2\bar{z}_2}\right]^{(1/2)[\cosh(2\lambda_2)+\sinh(2\lambda_2)]} \tag{69}$$

which agrees with the result in [8]. Here the subscripts on the coordinates refer to the different light cone coordinates in the charge density and spin density sectors which are obtained when using the Luttinger model solution, i.e. $u_1 = \tilde{v}_{F1}\tau + i\sigma$ with $\tilde{v}_{F1} = v_F[1-4(h_{11}+h_{12})^2]^{1/2}$ etc. For the Thirring model solution there is no light cone renormalisation, and these two coordinates coincide. From these functions it is easy to see the phase portrait. The phase transition lines occur for $h_{11} = h_{12}$ and $h_{11} = -h_{12}$ independent of which solution is used. Comparing with the above results, we see that the point at which the enhanced $SU(3)_L \times SU(3)_R$ symmetry occurs in H' lies on these phase transition lines at the value $h_{11} + h_{12} = -\frac{1}{3}$. The result is shown in figure 1. We remark that it would be of interest to examine renormalisation group studies of actions which break the conformal invariance and have the $c = 2$ theory as fixed point. This might reveal additional information regarding the enhanced symmetry point and its possible relation to the phase structure.

5. Conclusion

We have seen how the Luttinger model may be mapped onto the Thirring model and the model for compactified bosons taking values in a torus. These latter models have peculiar enhanced symmetries for particular values of the radii that translate to enhanced symmetries for the Luttinger model at particular values of the couplings. For the Luttinger model with spin the relevant symmetry enhanced symmetry is $SU(3)_L \times SU(3)_R$, and the symmetry point is found to lie on the phase transition lines.

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